Topological Cyclic Homology

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Count with higher homotopy

Ordinals

$0 = \phi$, $1 = \{0\}$, $2 = \{0, 1\}, 3 = \{0, 1, 2\}$, $\ldots$

Distinguishable objects

$0, 1, 2, 3, 4, 5, \ldots$

Indistinguishable objects

$0, B\Sigma_1, B\Sigma_2, B\Sigma_3, \ldots$

$B\Sigma_n = E\Sigma_n/\Sigma_n$, where $E\Sigma_n$ is a space with free Σ_n action and non-equivariantly contractible.

Algebraic K theory

Let F be a field.

Classification of vector spaces

$rank = 0, 1, 2, 3, \ldots$

Moduli of vector spaces

$0, BGL_1(F), BGL_2(F), BGL_3(F), \ldots$

$GL_n(F)$ is the symmetry group of vector spaces of rank n.

$K(F)$ is the group completion of $\coprod BGL_n(F)$

Infinite loop spaces

- E_∞-space: space with a multiplication which is commutative and associative up to coherent higher homotopy
- E_∞-group: E_∞-space with a homotopy inverse
- group completion: $X \to \Omega BX$
- infinite loop space: $X \cong \Omega X_1 \cong \Omega^2 X_2 \cong \dots$
- delooping machine: E_∞-group $\Leftrightarrow$ infinite loop space
- examples of E_∞-spaces: $\coprod_{n\geq 0} B\Sigma_n$, $\coprod_{n\geq 0} BGL_n(F)$

Spectra

Spectrum

A sequence of spaces X_i with $X_i \cong \Omega X_{i+1}$

- spectrum $\Leftrightarrow$ generalized homology theory
- infinite loop space $\Leftrightarrow$ connective spectrum
- symmetric monoidal category under smash product
- $(\coprod_{n\geq 0} B\Sigma_n)_+ \cong \Omega^\infty \mathbb{S}$
- $K(F)$ is an E_∞ ring spectrum

Morita equivalence

Rings could have equivalent module categories without being isomorphic.

Example

For any ring R, the category of R-modules is equivalent to the category of $Mat_n(R)$-modules:

$$- \otimes_R R^n : R\text{-}Mod \rightleftarrows Mat_n(R)\text{-}Mod : - \otimes_{Mat_n(R)} R^n$$

- The functor $K(-)$ factors through the (derived) category of modules (as abelien categories/stable ∞-categories)
- $K(R) \cong K(Mat_n(R))$

Hochschild homology

Let R be an associative algebra over a field k.

Define Hochschild homology as

$$HH_*(R/k) = Tor_*^{R \otimes_k R^{op}}(R, R)$$

Theorem (Hochschild-Kostant-Rosenberg)

If R is commutative and smooth over k, $HH_*(R/k)$ is isomorphic to the algebra of differential forms $\Omega^*_{R/k}$

We also has a trace map

$$tr : K_0(R) \to HH_0(R/k)$$

by taking the trace of a projection.

Topological Hochschild homology

Let R be an associative ring (spectrum). We define

$$THH(R) = R \wedge_{R \wedge_{\mathbb{S}} R^{op}} R$$

Example(Bökstedt-Waldhausen)

For $R = \Sigma^\infty \Omega X_+$, we have

$$THH(R) \cong \Sigma^\infty LX_+$$

where LX is the free loop space.

The Dennis trace

There are maps

$$BGL_n(F) \to LBGL_n(F) \to \Omega^\infty THH(Mat_n(F))$$

- THH turns Morita equivalence to homotopy equivalences
- There is a topological Dennis trace map

$$K(F) \to THH(F)$$

Equivariant stable homotopy

Let G be a finite group. An G-equivariant stable type is characterized by the homotopy types of its fixed points by all subgroups of G.

Burnside category

- Object: finite G-sets
- Morphism category of $G\mathcal{E}$:
  groupoid of spans over G-sets A and B
- Morphism spectrum of $G\mathcal{A}$:
  algebraic K-theory spectrum of $Hom_{G\mathcal{E}}(A, B)$

The category of equivariant spectra is the category of spectrally enriched presheaves over $G\mathcal{A}$

Geometric fixed points

For a G-equivariant spectrum E, we can define the geometric fixed points $\Phi^G E$.

- $\Phi^G \Sigma^\infty_G X \cong \Sigma^\infty X^G$
- $\Phi^G(E_1 \wedge E_2) \cong \Phi^G E_1 \wedge \Phi^G E_2$
- Φ^G preserves homotopy colimits

Cyclotomic structure on THH

Theorem (Bökstedt-Hsiang-Madsen)

There is an S^1-equivariant equivalence

$$THH(R) \cong \Phi^{C_p} THH(R)$$

where S^1 acts on $\Phi^{C_p} THH(R)$ via the isomorphism $S^1 \cong S^1/C_p$

We say $THH(R)$ is a cyclotomic spectrum.

The Tate diagram

- homotopy fixed points: X^{hG}
- homotopy orbit: X_{hG}
- Tate spectrum: $X_{hG} \xrightarrow{N} X^{hG} \to X^{tG}$

Tate diagram

$$
\begin{array}{ccccc}
X_{hC_{p^n}} & \longrightarrow & X^{C_{p^n}} & \longrightarrow & (\Phi^{C_p} X)^{C_{p^{n-1}}} \\
\downarrow & & \downarrow & & \downarrow \\
X_{hC_{p^n}} & \longrightarrow & X^{hC_{p^n}} & \longrightarrow & X^{tC_{p^n}}
\end{array}
$$

The Frobenius

(Nikolaus-Scholze)

The data of a cyclotomic spectrum is equivalent to a naive S^1-equivariant spectrum X together with a S^1-equivariant Frobenius

$$\varphi : X \to X^{tC_p}$$

Example (Nikolaus-Scholze)

If A is an E_∞ ring spectrum (e.g. commutative ring), the Frobenius is the unique S^1-equivariant E_∞ map making the following diagram commute

$$\begin{array}{ccc} A & \longrightarrow & THH(A) \\ \downarrow{\scriptstyle \Delta_p} & & \downarrow{\scriptstyle \varphi} \\ (A \wedge \cdots \wedge A)^{tC_p} & \longrightarrow & THH(A)^{tC_p} \end{array}$$

Topological cyclic homology

- $TR^n(R) = THH(R)^{C_{p^n}}$
- $TR(R) = lim_R TR^n(R)$
- $TF(R) = lim_F TR^n(R)$
- $TC^-(R) = THH(R)^{hS^1}$
- $TP(R) = THH(R)^{tS^1}$
- $TC(R) = fib(TC^-(R) \xrightarrow{can-\varphi} TP(R)) \cong lim_{R,F} TR^n(R)$

The cyclotomic trace

The Dennis trace map lifts to topological cyclic homology:

$$tr : K(R) \to TC(R)$$

(Dundas-Goodwillie-McCarthy)

Suppose that $I \subseteq R$ is a nilpotent ideal. Then the diagram

$$\begin{array}{ccc} K(R) & \xrightarrow{tr} & TC(R) \\ \downarrow & & \downarrow \\ K(R/I) & \xrightarrow{tr} & TC(R/I) \end{array}$$

is homotopy cartesian (after p-completion).

Bökstedt periodicity

- $THH_*(\mathbb{F}_p) = \mathbb{F}_p[u]$, $|u| = 2$
- $TR^n_*(\mathbb{F}_p) = \mathbb{Z}/p^n[u]$
- $TF_*(\mathbb{F}_p) = \mathbb{Z}_p[u]$
- $TC^-_*(\mathbb{F}_p) = \mathbb{Z}_p[u, v]/(uv - p)$, $|v| = -2$
- $TP_*(\mathbb{F}_p) = \mathbb{Z}_p[\sigma^{\pm}]$, $|\sigma| = 2$
- $TC_*(\mathbb{F}_p) = \mathbb{Z}_p\{1, \zeta\}$, $|\zeta| = -1$

Relative THH

Let K be a local field of characteristic 0, with ring of integers $\mathcal{O}_K$, uniformizer ϖ and residue field k of characteristic p.
Define relative THH as

$$THH(\mathcal{O}_K/\mathbb{S}[z]) = \mathcal{O}_K \otimes_{\mathcal{O}_K \otimes_{\mathbb{S}[z]} \mathcal{O}_K^{op}} \mathcal{O}_K$$

where $\mathbb{S}[z] = \Sigma^\infty \mathbb{N}$, and we view $\mathcal{O}_K$ as an $\mathbb{S}[z]$-algebra by sending z to ϖ.

Relative TC

Let $W(k)$ be the ring of Witt vectors for k, then we have

$$\mathcal{O}_K \cong W(k)[z]/(E(z))$$

for some Eisenstein polynomial $E(z)$ with constant term p.

- $THH_*(\mathcal{O}_K/\mathbb{S}[z]) \cong \mathcal{O}_K[u]$
- $TP_*(\mathcal{O}_K/\mathbb{S}[z]) \cong W(k)[z, \sigma^{\pm}]$
- $TC^-(\mathcal{O}_K/\mathbb{S}[z]) \cong W(k)[z, u, v]/(uv - E(z))$

Resolution of $THH(\mathcal{O}_K)$

We have a cosimplicial resolution

$$THH(\mathcal{O}_K) \to THH(\mathcal{O}_K/\mathbb{S}[z_0]) \to THH(\mathcal{O}_K/\mathbb{S}[z_0, z_1]) \to \dots$$

This is an analogue of the Adams resolution in homotopy theory, and the Čech cover in algebraic geometry.

(Liu-Wang)

The totalization of the cosimplicial cyclotomic spectrum $THH(\mathcal{O}_K/\mathbb{S}[z]^{\wedge\bullet})$ is equivalent to $THH(\mathcal{O}_K)$ (after p-completion).

The descent spectral sequence

We have spectral sequences

- $THH_*(\mathcal{O}_K/\mathbb{S}[z]^{\wedge\bullet}) \Rightarrow THH_*(\mathcal{O}_K)$
- $TC_*^-(\mathcal{O}_K/\mathbb{S}[z]^{\wedge\bullet}) \Rightarrow TC_*^-(\mathcal{O}_K)$
- $TP_*(\mathcal{O}_K/\mathbb{S}[z]^{\wedge\bullet}) \Rightarrow TP_*(\mathcal{O}_K)$

(Liu-Wang)

The E_2 terms of the descent spectral sequence for $TP(\mathcal{O}_K)$ can be computed by the Ext groups of the Hopf algebroid

$$(TP_0(\mathcal{O}_K/\mathbb{S}[z]), TP_0(\mathcal{O}_K/\mathbb{S}[z_0, z_1]))$$

The structure of $TP_0(\mathcal{O}_K/\mathbb{S}[z_0, z_1])$

A δ-ring structure on a torsion-free commutative ring R is a lift of the Frobenius of R/p to R.

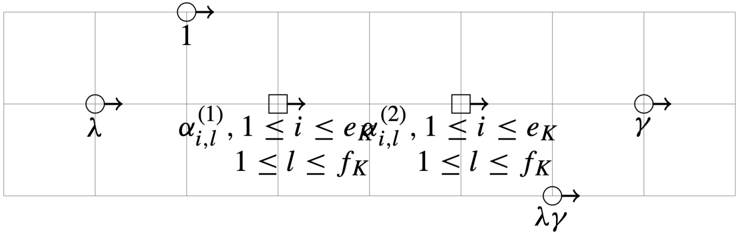
(Liu-Wang)

$TP_0(\mathcal{O}_K/\mathbb{S}[z_0, z_1])$ is the completion (under the Nygaard filtration) of the δ-ring generated by

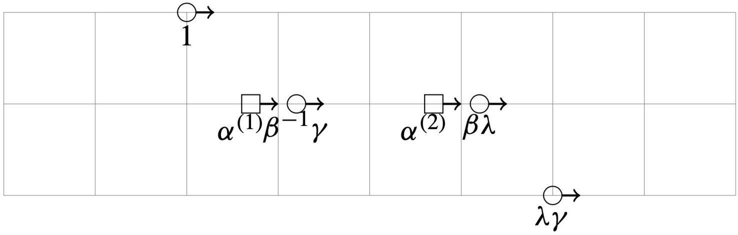
$$h = \frac{\varphi(z_0 - z_1)}{\varphi(E(z_0))}$$

over the δ-ring $W(k)[z_0, z_1]$.

The descent spectral sequence for $TC(\mathcal{O}_K; \mathbb{F}_p)$



The motivic spectral sequence for $K(K;\mathbb{F}_p)$



Thanks!